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QUALITATIVE INVESTIGATION OF THE STRESS-STRAIN STATE OF A SANDWICH PLATE

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The problem of the passage to the limit from three-dimensional problems of elasticity theory to two-dimensional problems has been investigated in [1, 2] for multilayered plates. A first iterarion process has been constructed therein on the basis of methods developed in [3, 4].

A construction of homogeneous solutions of elasticity theory problems for sandwich plates of symmetric configuration is given below. As in the case of a homogeneous plate [6], it is shown that the complete solution consists of a biharmonic, potential and vortex solution. The potential and vortex solutions are in the nature of an edge effect. Comparing them to the case of a homogeneous plate, shows that the edge effects can be both weaker and stronger, depending on the physical and geometric parameters of the sandwich plate.

The accuracy of some applied theories [6] is analyzed on the basis of the solution constructed, and limits for their applicability are established.

1. Let us consider a sandwich plate comprised of isotropic layers which are symmetric relative to the middle plane of the middle layer (Fig. 1). Let μ_i denote the shear modulus, i the number of the layer, σ_i the Poisson's ratio. Let the outer layers of thickness δ have the elastic characteristics ν_1 and μ_1, and the inner layer of thickness $2h$ the elastic characteristics ν_2 and μ_2.

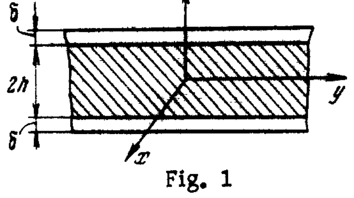

Fig. 1

Let us assume the outer plane faces to be stressfree, i. e. let us examine the homogeneous solutions. Let us utilize the method proposed in [5, 7] to construct the solution. By satisfying the conditions of contiguity of the displacements and stresses where the layers are connected, and the homogeneity conditions on the outer faces, we obtain three kinds of stress-strain states of a sandwich plate: a biharmonic, potential and vortical.

The biharmonic solution is

$$u_2^{(1)} = \lambda a\left[(\nu_2+1)\zeta\partial\psi/\partial\xi - {}^1/_2(\nu_2+{}^1/_3)\zeta^3\lambda^2\partial\Delta\psi/\partial\xi\right] \tag{1.1}$$
$$v_2^{(1)} = \lambda a\left[(\nu_2+1)\zeta\partial\psi/\partial\eta - {}^1/_2(\nu_2+{}^1/_3)\zeta^3\lambda^2\partial\Delta\psi/\partial\eta\right]$$
$$w_2^{(1)} = -(\nu_2+1)a\psi + a\lambda^2\left[2\nu_2 f_1(0) - {}^1/_2(\nu_2-1)\zeta^2\right]\Delta\psi$$
$$\sigma_{z,2}^{(1)} = 0 \qquad (\nu_i = 1/(1-2\sigma_i))$$

$$\tau_{xz,2}^{(1)} = 2\mu_2\nu_2\lambda^2 f_1\partial\Delta\psi/\partial\xi, \qquad \tau_{yz,2}^{(1)} = 2\mu_2\nu_2\lambda^2 f_1\partial\Delta\psi/\partial\eta$$
$$\sigma_{x,2}^{(1)} = 2\mu_2\lambda\left\{\left[2\nu_2\frac{\partial^2\psi}{\partial\xi^2} + (\nu_2-1)\frac{\partial^2\psi}{\partial\eta^2}\right]\zeta - \left(\nu_2+\frac{1}{3}\right)\frac{\zeta^3}{2}\lambda^2\frac{\partial^2\Delta\psi}{\partial\xi^2}\right\}$$
$$\sigma_{y,2}^{(1)} = 2\mu_2\lambda\left\{\left[2\nu_2\frac{\partial^2\psi}{\partial\eta^2} + (\nu_2-1)\frac{\partial^2\psi}{\partial\xi^2}\right]\zeta - \left(\nu_2+\frac{1}{3}\right)\frac{\zeta^3}{2}\lambda^2\frac{\partial^2\Delta\psi}{\partial\eta^2}\right\}$$
$$\tau_{xy,2}^{(1)} = 2\mu_2\lambda\left[(\nu_2+1)\zeta\frac{\partial^2\psi}{\partial\xi\partial\eta} - (\nu_2+{}^1/_3)\frac{\zeta^3}{2}\lambda^2\frac{\partial^2\Delta\psi}{\partial\xi\partial\eta}\right] \tag{1.2}$$
$$u_1^{(1)} = u_2^{(1)} + a\lambda^3 f_2\partial\Delta\psi/\partial\xi, \qquad v_1^{(1)} = v_2^{(1)} + a\lambda^3 f_2\partial\Delta\psi/\partial\eta$$
$$w_1^{(1)} = w_2^{(1)} + a\lambda^2\theta_2(\zeta^2-\lambda_1^2)\Delta\psi, \qquad \sigma_{z,1}^{(1)} = 0$$
$$\tau_{xz,1}^{(1)} = 2\mu_1\nu_1\theta_1(1-\zeta^2)\lambda^2\partial\Delta\psi/\partial\xi$$
$$\tau_{yz,1}^{(1)} = 2\mu_1\nu_1\theta_1(1-\zeta^2)\lambda^2\partial\Delta\psi/\partial\eta$$
$$\sigma_{x,1}^{(1)} = p^{-1}\sigma_{x,2}^{(1)} + 2\mu_1(-2\theta_2\zeta\lambda\Delta\psi + \lambda^3 f_2\partial^2\Delta\psi/\partial\xi^2)$$
$$\sigma_{y,1}^{(1)} = p^{-1}\sigma_{y,2}^{(1)} + 2\mu_1(-2\theta_2\zeta\lambda\Delta\psi + \lambda^3 f_2\partial^2\Delta\psi/\partial\eta^2)$$
$$\tau_{xy,1}^{(1)} = p^{-1}\tau_{xy,2}^{(1)} + 2\mu_1\lambda^3 f_2\partial^2\Delta\psi/\partial\xi\partial\eta$$
$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{a}, \quad \zeta = \frac{z}{h+\delta}, \quad \lambda = \frac{h+\delta}{a}, \quad \lambda_1 = \frac{h}{h+\delta}$$
$$\lambda_2 = \frac{\delta}{h+\delta}, \quad p = \frac{\mu_2}{\mu_1}, \quad \theta_1 = \frac{1+\nu_2}{1+\nu_1}, \quad \theta_2 = \frac{\nu_2-\nu_1}{1+\nu_1}$$
$$f_1 = \lambda_1^2 + p^{-1}\nu_1\nu_2^{-1}\theta_1(1-\lambda_1^2) - \zeta^2$$
$$f_2 = \theta_2({}^1/_3\zeta^3 - \zeta\lambda_1^2 + {}^2/_3\lambda_1^3) - 2\nu_1\theta_1(1-p^{-1})(1-\lambda_1^2)(\zeta-\lambda_1)$$

Here a is the characteristic linear dimension of the sandwich plate in the xy plane, Δ the Laplace operator, ψ some biharmonic function of the variables ζ, η.

It is seen from (1.2) that for small λ and large p the displacements $u_1^{(1)}$. $v_1^{(1)}$ are determined by the displacements $u_2^{(1)}$, $v_2^{(1)}$ to the accuracy of terms of order λ^3. If $p \sim \lambda^2$, then the corrections become on the order of the first term. But the quantity $w_1^{(1)}$ is determined by $w_2^{(1)}$ to the accuracy of terms on the order of λ^2 and the corrections are independent of p.

The potential solution is of the form (here and henceforth, the summation over k is between 1 and ∞)

$$u_i^{(2)} = \lambda a \Sigma a_{ik} \frac{\partial C_k}{\partial \xi}, \quad v_i^{(2)} = \lambda a \Sigma a_{ik} \frac{\partial C_k}{\partial \eta}, \quad w_i^{(2)} = a \Sigma b_{ik} C_k \tag{1.3}$$

$$\tau_{xz,i}^{(2)} = 2\mu_i \Sigma r_{ik} \frac{\partial C_k}{\partial \xi}, \qquad \tau_{yz,i}^{(2)} = 2\mu_i \Sigma r_{ik} \frac{\partial C_k}{\partial \eta}$$

$$\sigma_{x,i}^{(2)} = 2\mu_i \left[\frac{\nu_i - 1}{\lambda} \Sigma s_{ik} C_k + \lambda \Sigma a_{ik} \frac{\partial^2 C_k}{\partial \xi^2}\right]$$

$$\sigma_{y,i}^{(2)} = 2\mu_i \left[\frac{\nu_i - 1}{\lambda} \Sigma s_{ik} C_k + \lambda \Sigma a_{ik} \frac{\partial^2 C_k}{\partial \eta^2}\right]$$

$$\sigma_{z,i}^{(2)} = \frac{2\mu_i}{\lambda} \Sigma t_{ik} C_k, \qquad \tau_{xy,i}^{(2)} = 2\mu_i \lambda \Sigma a_{ik} \frac{\partial^2 C_k}{\partial \xi \partial \eta}$$

$$a_{2k} = E_{1k} \sin \gamma_k \zeta + E_{2k} \zeta \cos \gamma_k \zeta \tag{1.4}$$

$$b_{2k} = E_{1k} \gamma_k \cos \gamma_k \zeta - E_{2k} [\gamma_k \zeta \sin \gamma_k \zeta + (2 + \nu_2) \nu_2^{-1} \cos \gamma_k \zeta]$$

$$r_{2k} = E_{1k} \gamma_k \cos \gamma_k \zeta - E_{2k} [\gamma_k \zeta \sin \gamma_k \zeta + \nu_2^{-1} \cos \gamma_k \zeta]$$

$$t_{2k} = -\gamma_k^2 \{E_{1k} \sin \gamma_k \zeta + E_{2k} [\zeta \cos \gamma_k \zeta - (1 + \nu_2^{-1}) \gamma_k^{-1} \sin \gamma_k \zeta]\}$$

$$s_{2k} = E_{2k} \nu_2^{-1} \gamma_k \sin \gamma_k \zeta$$

$$a_{1k} = D_{1k} \sin (\zeta - \lambda_1) \gamma_k + D_{2k} \cos (\zeta - \lambda_1) \gamma_k + D_{3k} (\zeta - \lambda_1) \sin (\zeta - \lambda_1) \gamma_k + D_{4k} (\zeta - \lambda_1) \cos (\zeta - \lambda_1) \gamma_k$$

$$b_{1k} = D_{1k} \gamma_k \cos (\zeta - \lambda_1) \gamma_k - D_{2k} \gamma_k \sin (\zeta - \lambda_1) \gamma_k + D_{3k} [\gamma_k (\zeta - \lambda_1) \cos (\zeta - \lambda_1) \gamma_k - (1 + 2\nu_1^{-1}) \sin (\zeta - \lambda_1) \gamma_k] - D_{\varphi k} [\gamma_k (\zeta - \lambda_1) \sin (\zeta - \lambda_1) \gamma_k + (1 + 2\nu_1^{-1}) \cos (\zeta - \lambda_1) \gamma_k]$$

$$r_{1k} = D_{1k} \cos (\zeta - \lambda_1) \gamma_k - D_{2k} \sin (\zeta - \lambda_1) \gamma_k + D_{3k} [\gamma_k (\zeta - \lambda_1) \cos \gamma_k (\zeta - \lambda_1) - \nu_1^{-1} \sin (\zeta - \lambda_1) \gamma_k] - D_{4k} [\gamma_k (\zeta - \lambda_1) \sin (\zeta - \lambda_1) \gamma_k + \nu_1^{-1} \cos (\zeta - \lambda_1) \gamma_k]$$

$$t_{1k} = -\gamma_k^2 \{D_{1k} \sin (\zeta - \lambda_1) \gamma_k + D_{2k} \cos (\zeta - \lambda_1) \gamma_k + D_{3k} [(\zeta - \lambda_1) \sin (\zeta - \lambda_1) \gamma_k + (1 + \nu_1) (\nu_1 \gamma_k)^{-1} \cos (\zeta - \lambda_1) \gamma_k] + D_{4k} [(\zeta - \lambda_1) \cos (\zeta - \lambda_1) \gamma_k - (1 + \nu_1) (\nu_1 \gamma_k)^{-1} \sin (\zeta - \lambda_1) \gamma_k]\}$$

$$s_{1k} = \gamma_k \nu_1^{-1} [D_{4k} \sin (\zeta - \lambda_1) \gamma_k - D_{3k} \cos (\zeta - \lambda_1) \gamma_k]$$

$$E_{1k} = \frac{1 + \nu_1}{\nu_1} \Big\{p \frac{1 + \nu_1}{\nu_1} \cos \lambda_2 \gamma_k \left(\lambda_1 \sin \lambda_1 \gamma_k + \frac{\cos \lambda_1 \gamma_k}{\nu_1 \gamma_k}\right) + \lambda_1 \sin \lambda_2 \gamma_k \cos \lambda_1 \gamma_k (1 + p\nu_1^{-1}) + \lambda_1 \lambda_2 \gamma_k \cos (\lambda_1 - \lambda_2) \gamma_k (1 - p) - \frac{1}{\gamma_k} \frac{1 + \nu_2}{\nu_1 \nu_2} p \sin \lambda_1 \gamma_k \sin \lambda_2 \gamma_k - \frac{\lambda_2}{\nu_2} [\sin \lambda_2 \gamma_k \cos \lambda_1 \gamma_k (p - 2 - \nu_2) - p \gamma_k^{-1} (1 + \nu_2) \sin \lambda_1 \gamma_k \cos \lambda_2 \gamma_k]\Big\} \tag{1.5}$$

$$E_{2k} = \frac{1 + \nu_1}{\nu_1} \left[p \frac{1 + \nu_1}{\nu_1} \cos \lambda_1 \gamma_k \cos \lambda_2 \gamma_k - \left(1 + \frac{p}{\nu_1}\right) \sin \lambda_1 \gamma_k \sin \lambda_2 \gamma_k - \lambda_2 \gamma_k (1 - p) \sin (\lambda_1 - \lambda_2) \gamma_k\right]$$

(cont.)

$$D_{1k} = -\sin\lambda_2\gamma_k\left\{\frac{\lambda_1}{\nu_1^2}(1-2p-p\nu_2)(p+\nu_1)+\frac{\sin 2\lambda_1\gamma_k}{2\gamma_k}\left[p^2\frac{2+\nu_1}{\nu_1^2}+\right.\right.$$
$$\left.\left.+\frac{2+\nu_2}{\nu_1\nu_2}+p\frac{(1+\nu_1)^2}{\nu_2\nu_1}-p\frac{2+\nu_1}{\nu_1\nu_2}\right]\right\}+\frac{1+\nu_1}{\nu_1}\left(\lambda_2\sin\lambda_2\gamma_k+\frac{\cos\lambda_2\gamma_k}{\nu_1\gamma_k}\right)$$
$$p\frac{\nu_2+1}{\nu_2}\cos^2\lambda_1\gamma_k+\lambda_2\cos\lambda_2\gamma_k\left[\lambda_1\gamma_k(1-p)\left(p\frac{2+\nu_1}{\nu_1}-\frac{1}{\nu_1}\right)+\right.$$
$$\left.+\frac{1}{2}\sin 2\lambda_1\gamma_k\left(p^2\frac{2+\nu_1}{\nu_1}-\frac{2+\nu_2}{\nu_1\nu_2}+p\frac{3+\nu_1}{\nu_1\nu_2}\right)\right]$$

$$D_{2k} = \frac{1+\nu_1}{\nu_1}\left\{p\frac{1+\nu_1}{\nu_1}\cos\lambda_2\gamma_k\left(\lambda_1+\frac{\sin 2\lambda_1\gamma_k}{\nu_2\gamma_k}\right)+p\frac{1+\nu_2}{\nu_2}\sin^2\lambda_1\gamma_k\times\right.$$
$$\left.\times\left(\lambda_2\cos\lambda_2\gamma_k-\frac{\sin\lambda_2\gamma_k}{\nu_1\gamma_k}\right)+\lambda_2\sin\lambda_2\gamma_k\left[\lambda_1\gamma_k(1-p)-\frac{\sin 2\lambda_1\gamma_k}{2\nu_2}(p-2-\nu_2)\right]\right\}$$

$$D_{3k} = p\frac{(1+\nu_2)(1+\nu_1)}{\nu_2\nu_1}\sin\lambda_2\gamma_k\sin^2\lambda_1\gamma_k-$$
$$-p\frac{1+\nu_1}{\nu_1}\cos\lambda_2\gamma_k\left[\lambda_1\gamma_k(1-p)+{}^1/_2\sin 2\lambda_1\gamma_k(p+\nu_2^{-1})\right]-$$
$$-\lambda_2\gamma_k\sin\lambda_2\gamma_k(1-p)\left[\lambda_1\gamma_k(1-p)+{}^1/_2\sin 2\lambda_1\gamma_k(p+1+2\nu_2^{-1})\right]$$

$$D_{4k} = p\frac{(1+\nu_2)(1+\nu_1)}{\nu_2\nu_1}\cos\lambda_2\gamma_k\cos^2\lambda_1\gamma_k-\sin\lambda_2\gamma_k\left[\lambda_1\gamma_k(1-p)(1+p\nu_1^{-1})+\right.$$
$$\left.+\frac{\sin 2\lambda_1\gamma_k}{2\nu_1\nu_2}(2\nu_1+\nu_2\nu_1+p^2\nu_2-p\nu_1+p)\right]-\lambda_2\gamma_k\cos\lambda_2\gamma_k(1-p)\left[\lambda_1\gamma_k(1-p)+\right.$$
$$\left.+{}^1/_2\sin 2\lambda_1\gamma_k(1+p+2\nu_2^{-1})\right]$$

The function $C_k(\zeta, \eta)$ are determined from the equations

$$\frac{\partial^2 C_k}{\partial\xi^2}+\frac{\partial^2 C_k}{\partial\eta^2}-\frac{\gamma_k^2}{\lambda^2}C_k=0 \tag{1.6}$$

The letters γ_k denote nonzero roots of the function

$$F(\gamma) = (p-1)(\gamma^2\lambda_2^2-\sin^2\gamma\lambda_2)\left[2\gamma\lambda_1+(1+2\nu_2^{-1})\sin 2\gamma\lambda_1\right]- \tag{1.7}$$
$$-p(p-1)(2\gamma\lambda_1-\sin 2\gamma\lambda_1)\left[\gamma^2\lambda_2^2-\nu_1^{-2}-(1+2\nu_1^{-1})\cos^2\gamma\lambda_2\right]+$$
$$+p\ (1+\nu_1^{-1})\left[(\nu_1-\nu_2)(\nu_1\nu_2)^{-1}(2\gamma\lambda_1-\sin 2\gamma\lambda_1)+\right.$$
$$\left.+(1+\nu_2^{-1})(2\gamma-\sin 2\gamma)\right]=0$$

The vortex solution has the form

$$u_i^{(3)} = 2\lambda^2 a\Sigma l_{k,i}\frac{\partial B_k}{\partial\eta},\quad v_i^{(3)} = -2\lambda^2 a\Sigma l_{k,i}\frac{\partial B_k}{\partial\xi},\quad w_i^{(3)}=0 \tag{1.8}$$
$$\tau_{xz,i}^{(3)} = 2\mu_i\lambda\Sigma l'_{k,i}\frac{\partial B_k}{\partial\eta},\quad \tau_{yz,i}^{(3)} = -2\mu_i\lambda\Sigma l'_{k,i}\frac{\partial B_k}{\partial\xi},\quad \sigma_{z,i}^{(3)}=0$$
$$\tau_{xy,i}^{(3)} = 2\mu_i\lambda^2\Sigma l_{k,i}\left(\frac{\partial^2 B_k}{\partial\eta^2}-\frac{\partial^2 B_k}{\partial\xi^2}\right),\quad \sigma_{x,i}^{(3)} = -\sigma_{y,i}^{(3)} = 4\mu_i\lambda^2\Sigma l_{k,i}\frac{\partial^2 B_k}{\partial\xi\partial\eta}$$

$$l_{k,2} = \beta_k^{-1}\sin\beta_k\zeta \tag{1.9}$$
$$l_{k,1} = \beta_k^{-1}p\cos\lambda_1\beta_k\sin(\zeta-\lambda_1)\beta_k+\beta_k^{-1}\sin\lambda_1\beta_k\cos(\zeta-\lambda_1)\beta_k$$

The functions $B_k(\xi, \eta)$ are determined from the equation

$$\frac{\partial^2 B_k}{\partial \xi^2} + \frac{\partial^2 B_k}{\partial \eta^2} - \frac{\beta_k^2}{\lambda^2} B_k = 0 \qquad (1.10)$$

The quantity β_k is the root of the equation

$$(1 + p) \cos \beta = (1 - p) \cos (2\lambda_1 - 1) \beta \qquad (1.11)$$

2. Let us show that (1.11) has only real roots. Let us use the notation

$$k = (1 - p)(1 + p)^{-1}, \qquad \lambda_3 = 2\lambda_1 - 1$$

Since $p \in (0, \infty)$, $\lambda_1 \in (0, 1)$, then $k \in (-1, 1)$, $\lambda_3 \in (-1, 1)$. Let us assume that (1.11) has the complex root $\beta = x + iy$. Separating real and imaginary parts, we have

$$\cos x \operatorname{ch} y = k \cos \lambda_3 x \operatorname{ch} \lambda_3 y, \qquad \sin x \operatorname{sh} y = k \sin \lambda_3 x \operatorname{sh} \lambda_3 y$$

Hence

$$1 = k^2 \left[\cos^2 \lambda_3 x \left(\frac{\operatorname{ch} \lambda_3 y}{\operatorname{ch} y}\right)^2 + \sin^2 \lambda_3 x \left(\frac{\operatorname{sh} \lambda_3 y}{\operatorname{sh} y}\right)^2\right] \qquad (2.1)$$

Since

$$(\operatorname{ch} \lambda_3 y / \operatorname{ch} y)^2 < 1, \qquad (\operatorname{sh} \lambda_3 y / \operatorname{sh} y)^2 < 1$$

then we can write

$$1 < k^2 (\cos^2 \lambda_3 x + \sin^2 \lambda_3 x) = k^2$$

But $|k| < 1$, which results in a contradiction. It is thereby shown that Eq. (1.11) has no complex roots. Let us assume that Eq. (1.11) has the imaginary root $\beta = i\alpha$, then it becomes

$$\operatorname{ch} \alpha = k \operatorname{ch} \lambda_3 \alpha \qquad (2.2)$$

but because $|k| < 1$ and $|\lambda_3| < 1$, Eq. (2.2) has no real roots, and therefore, Eq. (1.11) has no imaginary roots.

Let us examine how β_1, the first positive root of (1.11), behaves. Since (1.8) is a boundary-layer type solution, the nature of the penetration of the solution within the region is determined by β_1. In the case of a homogeneous plate (1.11) is of the form

$$\cos \beta = 0 \qquad (2.3)$$

It is easy to see that $\beta_1 < {}^1/_2\pi$ for $p < 1$, i.e. for a weaker middle layer, and β_1 tends to zero as $p \to 0$. Therefore, in a three-layer sandwich plate with a weak middle layer the vortex solution penetrates more strongly within the domain than in a homogeneous plate, and the nature of the penetration will be stronger, the weaker the inner layer. If $p > 1$, then $\beta_1 > {}^1/_2\pi$ and the quantity β_1 tends to π as $p \to \infty$. Therefore, in the case of a rigid filler the vortex solution of a sandwich plate penetrates more weakly within the region than in a homogeneous plate, and the nature of the penetration is weaker, the more rigid the filler.

Let us examine (1.7). It can be shown that this equation has no imaginary roots, whereupon it follows that the potential solution is a boundary-layer type solution. Let us expand $F(\gamma)$ in a series in γ

$$F(\gamma) = \gamma^4 \, {}^4/_3 (1 + \nu_1^{-1}) \, p \, [p (1 + \nu_1^{-1}) \lambda_1^3 + (1 + \nu_2^{-1})(1 - \lambda_1^3)] + \ldots \qquad (2.4)$$

It is seen from (2.4) that the coefficient of γ^4 does not vanish for any changes in the parameters, therefore, there are no additional zero roots in (1.7).

Following [8], let us find the asymptotic roots of Eq. (1.7) in the first quadrant

$$\gamma_k = \operatorname{ctg} \varphi \, [\pi (2k - {}^1/_2) + \arg Z] + i \ln |Z^{-1} 2\pi k \operatorname{ctg} \varphi| \qquad (2.5)$$

In the first quadrant Eq. (1.7) has three branches of asymptotic roots corresponding to the three values of Z and $\operatorname{ctg}\varphi$. The asymptotics of the roots hence depends on λ_1

a) $$\lambda_1 < {}^1/_3, \qquad \operatorname{ctg}\varphi_1 = {}^1/_2\lambda_1^{-1} \tag{2.6}$$

$$Z_1 = \frac{p\,(1+\nu_1^{-1})\,(1+\nu_2^{-1}) - {}^1/_4\,(p-1)\,[1+2\nu_2^{-1} - (1+2\nu_1^{-1})\,p]}{\lambda_1\,(1-p)\,[1+p\,(1+2\nu_1^{-1})]}$$

$$\operatorname{ctg}\varphi_{2,3} = \frac{1}{1-\lambda_1}, \qquad Z_{2,3} = \pm\frac{1}{1-\lambda_1}\left[\frac{1+p\,(1+2\nu_1^{-1})}{2\,(1-p)}\right]^{1/2}$$

b) $$\lambda_1 = {}^1/_3, \qquad \operatorname{ctg}\varphi_{1\,2,3} = {}^3/_2 \tag{2.7}$$

The values of Z_1, Z_2, Z_3 are determined from the equation

$$16Z^3\,(1-p)^2 + 12Z^2\,(p-1)\,(p+1+2\nu_2^{-1}) + 18Z\,(p-1)\,[1+p\,(1+ \\ +2\nu_1^{-1})] + 27p\,(1+\nu_1^{-1})\,(1+\nu_2^{-1}) - {}^{27}/_4\,(p-1)\,[1+2\nu_2^{-1} - \\ - p\,(1+2\nu_1^{-1})] = 0$$

c) $$\lambda_1 > {}^1/_3, \qquad \operatorname{ctg}\varphi_{1,2} = (1-\lambda_1)^{-1} \tag{2.8}$$

$$Z_{1,2} = \pm\frac{1}{1-\lambda_1}\left[\frac{p\,(1+\nu_1^{-1})\,(1+\nu_2^{-1}) - {}^1/_4\,(p-1)\,[1+2\nu_2^{-1} - p\,(1+2\nu_1^{-1})]}{(1-p)\,[1+p\,(1+2\nu_1^{-1})]}\right]^{1/2}$$

$$\operatorname{ctg}\varphi_3 = {}^1/_2\lambda_1^{-1}, \qquad Z_3 = \frac{1+p\,(1+2\nu_1^{-1})}{\lambda_1\,(1-p)}$$

3. The hypothesis of shear stresses, whereby $\tau_{xz,\,2}$ and $\tau_{yz,\,2}$ are considered constant over the thickness of the middle layer, is often used in applied theories. Let us investigate the limits of applicability of this hypothesis. We find from (1.1)

$$\tau^{(1)}_{xz,2\,\max} = 2\mu_2\nu_2\lambda^2\,[\lambda_1^2 + \nu_1\,(p\nu_2)^{-1}\,\theta_1\,(1-\lambda_1^2)]\,\partial\Delta\psi/\partial\xi \tag{3.1}$$

$$\tau^{(1)}_{xz,2\,\min} = 2\mu_2\nu_1\lambda^2\nu_1(p\nu_2)^{-1}\theta_1\,(1-\lambda_1^2)\,\partial\Delta\psi/\partial\xi \tag{3.2}$$

Let us consider the ratio

$$\frac{\tau^{(1)}_{xz,\,2\max}}{\tau^{(1)}_{xz,\,2\min}} = 1 + \frac{p\nu_2}{\theta_1\nu_1}\,\frac{\lambda_1^2}{1-\lambda_1^2} \tag{3.3}$$

For the hypothesis of shear stresses to be satisfied it is necessary that

$$\frac{p\nu_2}{\theta_1\nu_1}\,\frac{\lambda_1^2}{1-\lambda_1^2} < \varepsilon \tag{3.4}$$

where ε is a previously assigned number characterizing the accuracy of the hypothesis. The inequality (3.4) depends on the dimensionless thickness λ_1 of the middle layer as well as on the ratio between the elastic moduli p. The inequality (3.4) depends slightly on Poisson's ratios since for any σ_i

$${}^1/_2 \leqslant \nu_2\,(\nu_1\theta_1)^{-1} \leqslant 2 \tag{3.5}$$

It is seen from (3.4) that the smaller the λ_1, the wider do the boundaries of applicability of this hypothesis extend. For any real materials a λ_1 can be found for which the hypothesis becomes valid, and conversely, a λ_1 can always be found for which the hypothesis will not be valid.

As an illustration, let us consider several examples. Let us prescribe ε and λ_1 and let us find p from (3.4). For simplicity, let us consider materials with equal Poisson's ratios.

Let us assume the accuracy to be 3% for the hypothesis of shear stresses, therefore $\varepsilon = 0.03$. Let us present values of p calculated from (3.4) for some values of λ_1

$\lambda_1 = 0.05$	0.1	0.3	0.5	0.7	0.9	0.95
$p < 12$	3	$3 \cdot 10^{-1}$	$9 \cdot 10^{-2}$	$3 \cdot 10^{-2}$	$7 \cdot 10^{-3}$	$3.3 \cdot 10^{-3}$

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ON THE STABILITY OF COMPRESSED BARS

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The problem of the stability of an incompressible elastic bar of variable stiffness, compressed along the axis, is considered. The validity of linearization is proved, and the equilibrium modes after buckling are investigated.

After reduction of the appropriate boundary value problem to an equation with a completely continuous operator, a theorem of Krasnosel'skii [1] on bifurcation can be applied. In utilizing this theorem the proof of the simplicity (or odd multiplicity) of the eigenvalue of the corresponding linearized problem is the principal difficulty.

The case of hinged supports of the bar ends was considered in [2]. In this case the linearized equation is of second order, and the simplicity of the eigenvalues results from